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On nonlinear interactions in a spectrum of inviscid gravity–capillary surface waves

By DENNIS HOLLIDAY

R. & D. Associates, P.O. Box 9695, Marina del Rey, California 90291

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Using the Fourier components of the velocity potential at the surface in an iterative perturbation theory, the author derives expressions for the modal frequency shifts and modal decay rates in a spectrum of inviscid gravity–capillary surface waves. The expressions differ from those calculated by the perturbation methods of Hasselmann (1962) and Valenzuela & Laing (1972), which are based on expanding the velocity potential about the equilibrium fluid level $z = 0$. This difference is shown to be due to the failure of the $z = 0$ perturbation method to converge rapidly enough to produce lowest-order corrections that are smaller than unperturbed quantities. A physical explanation for this failure is given.

1. Introduction

Hasselmann (1962) and Valenzuela & Laing (1972) have used perturbation theory to derive expressions for the modal decay rates in a spectrum of gravity waves and gravity–capillary waves, respectively. These derivations are based on retaining the first few terms in an expansion of the fluid-velocity potential about the equilibrium fluid level $z = 0$. This expansion is equivalent to approximating $e^{k\eta}$, where k is the wavenumber of a surface wave and η is the height of the surface above $z = 0$, by the first few terms in a Taylor series about $\eta = 0$:

$$e^{k\eta} \simeq 1 + k\eta + \tfrac{1}{2}k^2\eta^2 + \tfrac{1}{6}k^3\eta^3 + \tfrac{1}{24}k^4\eta^4. \tag{1.1}$$

Such an approximation is useful if $k\eta$ is not too large. In the case of a spectrum of waves the condition for the accuracy of (1.1) becomes†

$$\tfrac{1}{2}k^2\langle\eta^2\rangle \leqslant 1, \tag{1.2}$$

which was explicitly recognized by Phillips (1960) as limiting the validity of his analysis of nonlinear interactions among a quartet of discrete gravity wave modes. If the mean value $\langle\eta^2\rangle$ is calculated for wind-driven gravity waves in equilibrium, we obtain (Phillips 1969, chap. 4)

$$\tfrac{1}{2}k^2\langle\eta^2\rangle = \tfrac{1}{4}k^2B(U^2/g)^2. \tag{1.3}$$

In this equation $B \simeq 4{\cdot}6\times10^{-3}$, U = wind speed and $g = 9{\cdot}8\,\mathrm{m/s^2}$. For a 1m gravity wave and a 30 knot wind, we have from (1.3)

$$\tfrac{1}{2}k^2\langle\eta^2\rangle = 26{\cdot}9, \tag{1.4}$$

† Since the probability distribution of η is, to lowest order, Gaussian, it follows that the mean value $\langle\exp(k\eta)\rangle = \exp[\tfrac{1}{2}k^2\langle\eta^2\rangle]$, and exp [1] is well approximated by the first four terms of its Taylor series.

which violates (1.2). Furthermore, if k is the wavenumber of a capillary wave, the quantity $\frac{1}{2}k^2\langle\eta^2\rangle$ can easily be greater than 1000.

The examples above suggest that perturbation methods based on the $z = 0$ expansion of the velocity potential may not converge rapidly enough to produce lowest-order quantities that are small deviations from unperturbed quantities. To determine the validity of this conjecture we need to calculate, using the $z = 0$ method, some quantity that has a non-zero unperturbed value. The modal decay rate will not do, for its value is zero in the unperturbed linear case. An appropriate quantity appears to be the modal frequency, which for gravity–capillary waves has an unperturbed value $\omega_0(k) = (gk + Tk^3)^{\frac{1}{2}}$, where T is the surface tension per unit density.

In this paper expressions for both the modal frequency shift and the modal decay rate will be calculated by two different perturbation methods: the $z = 0$ expansion method of Hasselmann (1962) and Valenzuela & Laing (1972) and a method that uses the Fourier components of the velocity potential evaluated on the fluid surface $z = \eta(x,y,t)$. Section 2 of the paper contains a review of the modal equations and a derivation of a relationship between the $z = 0$ and $z = \eta(x,y,t)$ perturbation methods. In § 3, the modal equations are solved by iteration. Section 4 contains the final steps leading to the derivation of both $z = 0$ and $z = \eta$ expressions for the modal frequency shifts and modal decay rates. A comparison of the $z = 0$ and $z = \eta$ methods is made in § 5.

2. Basic modal equations

The velocity potential $\phi(x,y,z,t)$ and surface deformation $z = \eta(x,y,t)$ describing inviscid wave motion of an infinitely deep fluid obey the equations (Landau & Lifshitz 1959, chaps. I and VIII)

$$\partial\phi/\partial z = D\eta/Dt, \tag{2.1}$$

$$\partial\phi/\partial t + \tfrac{1}{2}(\nabla\phi)^2 = -g\eta + T\nabla\,.\,\mathbf{s}, \tag{2.2}$$

$$\nabla^2\phi = 0, \tag{2.3}$$

where the derivatives of $\phi(x,y,z,t)$ in (2.1) and (2.2) are evaluated at

$$z = \eta(x,y,t), \tag{2.4}$$

and

$$\mathbf{s} = [1 + (\nabla\eta)^2]^{-\frac{1}{2}}\nabla\eta; \tag{2.5}$$

g is the gravitational acceleration, T is the surface tension and, for convenience, the density is set equal to one. The equilibrium level of the fluid is $z = 0$.

Both $\eta(x\;y,t)$ and $\phi(x,y,z,t)$ can be expanded in modes with ϕ satisfying (2.3):

$$\eta(x,y,t) = \left(\frac{2\pi}{L}\right)\sum_{\mathbf{k}} B_{\mathbf{k}}(t)\,e^{i\mathbf{k}.\mathbf{r}}, \tag{2.6}$$

$$\phi(x,y,z,t) = \left(\frac{2\pi}{L}\right)\sum_{\mathbf{k}} A_{\mathbf{k}}(t)\,e^{kz}\,e^{i\mathbf{k}.\mathbf{r}}, \tag{2.7}$$

where

$$\mathbf{k} = (2\pi/L)\,\mathbf{n}; \tag{2.8}$$

$\mathbf{n}$ is the two-dimensional number vector whose components are positive and negative integers and $\mathbf{r}$ is the two-dimensional horizontal position vector; the orthogonality relation with this box normalization is†

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dy\, e^{i(\mathbf{k}-\mathbf{l}).\mathbf{r}} = L^2 \delta_{\mathbf{k},\mathbf{l}}, \tag{2.9}$$

where

$$\delta_{\mathbf{k},\mathbf{l}} = \begin{Bmatrix} 1 & \text{if} & \mathbf{k} = \mathbf{l}, \\ 0 & \text{if} & \mathbf{k} \neq \mathbf{l}. \end{Bmatrix} \tag{2.10}$$

Substituting (2.6) and (2.7) into (2.1) and (2.2) and following Hasselmann (1962) and Valenzuela & Laing (1972) by expanding e^{kz} in (2.7) about $z = 0$, we obtain

$$dB_{\mathbf{k}}(t)/dt - kA_{\mathbf{k}}(t) = \Gamma_{\mathbf{k}}(t), \tag{2.11}$$

where

$$\begin{aligned} \Gamma_{\mathbf{k}}(t) = {} & \left(\frac{2\pi}{L}\right) \Sigma A_{\mathbf{k}_1} B_{\mathbf{k}_2} \frac{k_1}{\omega_0(k_1)} C_1(\mathbf{k}_1, \mathbf{k}_2; N)\, \delta^{(2)} \\ & + \left(\frac{2\pi}{L}\right)^2 \Sigma A_{\mathbf{k}_1} B_{\mathbf{k}_2} B_{\mathbf{k}_3} \frac{k_1}{\omega_0(k_1)} C_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; N)\, \delta^{(3)} + \text{higher-order terms}, \end{aligned} \tag{2.12}$$

and

$$dA_{\mathbf{k}}(t)/dt + k^{-1}\omega_0(k)^2 B_{\mathbf{k}}(t) = \Lambda_{\mathbf{k}}(t), \tag{2.13}$$

where

$$\omega_0(k)^2 = gk + Tk^3 \tag{2.14}$$

and

$$\begin{aligned} \Lambda_{\mathbf{k}}(t) = {} & \left(\frac{2\pi}{L}\right) \Sigma B_{\mathbf{k}_1} B_{\mathbf{k}_2} \frac{\omega_0(k)}{k} C_2(\mathbf{k}_1, \mathbf{k}_2; N)\, \delta^{(2)} \\ & + \left(\frac{2\pi}{L}\right) \Sigma A_{\mathbf{k}_1} A_{\mathbf{k}_2} \frac{\omega_0(k)}{k} \frac{k_1 k_2}{\omega_0(k_1)\, \omega_0(k_2)} C_3(\mathbf{k}_1, \mathbf{k}_2; N)\, \delta^{(2)} \\ & + \left(\frac{2\pi}{L}\right)^2 \Sigma B_{\mathbf{k}_1} B_{\mathbf{k}_2} B_{\mathbf{k}_3} \frac{\omega_0(k)}{k} C_5(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; N)\, \delta^{(3)} \\ & + \left(\frac{2\pi}{L}\right)^2 \Sigma A_{\mathbf{k}_1} A_{\mathbf{k}_2} B_{\mathbf{k}_3} \frac{\omega_0(k)}{k} \frac{k_1 k_2}{\omega_0(k_1)\, \omega_0(k_2)} C_6(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; N)\, \delta^{(3)} \\ & + \text{higher-order terms}; \end{aligned} \tag{2.15}$$

the sums are over the subscripted wavenumbers and

$$\delta^{(2)} \equiv \delta_{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2}, \quad \delta^{(3)} \equiv \delta_{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}.$$

Expressions for $C_1(\mathbf{k}_1, \mathbf{k}_2)$, $C_2(\mathbf{k}_1, \mathbf{k}_2)$, $C_3(\mathbf{k}_1, \mathbf{k}_2)$, $C_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, $C_5(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, and $C_6(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ are given in appendix A.

Equations (2.11)–(2.15) are equivalent to equations (2.12) and (2.13) in Valenzuela & Laing (1972) except for what appears to be a typographical error in their paper. This apparent error is discussed in appendix B.

Instead of using the velocity-potential coefficient $A_{\mathbf{k}}(t)$ defined by (2.7), we can use the coefficient $\tilde{A}_{\mathbf{k}}(t)$ defined by

$$\phi(x, y, z = \eta(x, y, t), t) = \left(\frac{2\pi}{L}\right) \sum_{\mathbf{k}} \tilde{A}_{\mathbf{k}}(t)\, e^{i\mathbf{k}.\mathbf{r}}. \tag{2.16}$$

† Box normalization is convenient in many problems involving interacting waves. Its most common use in theoretical physics is in quantum mechanics [see, for example, Schiff 1955, pp. 49–50]. L is a large length, eventually to be taken infinite.

The relationship between $\tilde{A}_{\mathbf{k}}(t)$ and $A_{\mathbf{k}}(t)$ is found by setting $z = \eta(x, y, t)$ in (2.7), expanding about $z = 0$ and using (2.6):

$$\tilde{A}_{\mathbf{k}}(t) = A_{\mathbf{k}}(t) + \left(\frac{2\pi}{L}\right) \Sigma k_1 A_{\mathbf{k}_1}(t) B_{\mathbf{k}_2}(t) \delta^{(2)} + \left(\frac{2\pi}{L}\right)^2 \Sigma \tfrac{1}{2} k_1^2 A_{\mathbf{k}_1}(t) B_{\mathbf{k}_2}(t) B_{\mathbf{k}_3}(t) \delta^{(3)} + \ldots, \tag{2.17}$$

which can be iterated to give

$$A_{\mathbf{k}}(t) = \tilde{A}_{\mathbf{k}}(t) - \left(\frac{2\pi}{L}\right) \Sigma k_1 \tilde{A}_{\mathbf{k}_1}(t) B_{\mathbf{k}_2}(t) \delta^{(2)} + \left(\frac{2\pi}{L}\right)^2 \Sigma [k_1 |\mathbf{k}_1 + \mathbf{k}_2| - \tfrac{1}{2} k_1^2] \tilde{A}_{\mathbf{k}_1}(t) B_{\mathbf{k}_2}(t) B_{\mathbf{k}_3}(t) \delta^{(3)} + \ldots. \tag{2.18}$$

Substituting (2.18) into (2.11)–(2.15), we obtain equations that are identical in form except that $A_{\mathbf{k}}$ is replaced by $\tilde{A}_{\mathbf{k}}$ and C_1, C_2, C_3, etc., are replaced by different functions $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$, etc., which are also given in appendix A. In the case $T = 0$, the new equations for $\tilde{A}_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are equivalent to equations (11), (15) and (16) of Watson & West (1975) except that these authors have additional terms T_w and T_u which represent the effects of wind and slowly varying surface currents. Apparently regarding their use of the Fourier components of the potential at the surface as a convenience, Watson & West (1975) state that their equations, except for the terms T_w and T_u, are equivalent to Hasselmann's (1962) equations, despite the fact that the coefficients $\tilde{C}_1$, $\tilde{C}_2$, etc., are functionally different from C_1, C_2, etc. As we shall see in § 5, where we compare the solution for $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ with the solution for $\tilde{A}_{\mathbf{k}}$ and $B_{\mathbf{k}}$, this assertion is not correct.

In the analysis to follow we shall solve (2.11)–(2.15) by an iterative perturbation technique. Since the forms of the equations for the two pairs ($A_{\mathbf{k}}$, $B_{\mathbf{k}}$) and ($\tilde{A}_{\mathbf{k}}$, $B_{\mathbf{k}}$) are identical except for the coefficients C_1, C_2, etc., the solution for the pair ($\tilde{A}_{\mathbf{k}}$, $B_{\mathbf{k}}$) is determined by substituting $\tilde{C}_1$ for C_1, $\tilde{C}_2$ for C_2, etc, in the solution for the pair ($A_{\mathbf{k}}$, $B_{\mathbf{k}}$).

3. Iterative solution of the modal equations

By introducing a new quantity

$$\xi_{\mathbf{k}}(t) = \tfrac{1}{2}\left[B_{\mathbf{k}}(t) + i\frac{k}{\omega_0(k)} A_{\mathbf{k}}(t)\right], \tag{3.1}$$

so that

$$B_{\mathbf{k}} = \xi_{\mathbf{k}} + \xi^*_{-\mathbf{k}} \tag{3.2}$$

and

$$A_{\mathbf{k}} = -i\frac{\omega_0(k)}{k}(\xi_{\mathbf{k}} - \xi^*_{-\mathbf{k}}), \tag{3.3}$$

we can combine (2.11) and (2.13) into the first-order differential equation

$$d\xi_{\mathbf{k}}/dt + i\omega_0(k)\,\xi_{\mathbf{k}} = F_{\mathbf{k}}, \tag{3.4}$$

where

$$F_{\mathbf{k}} = \tfrac{1}{2}\left[\Gamma_{\mathbf{k}} + i\frac{k}{\omega_0(k)}\Lambda_{\mathbf{k}}\right]. \tag{3.5}$$

Equation (3.4) can be rewritten as

$$\xi_{\mathbf{k}}(t) = \xi^{(1)}_{\mathbf{k}}(t) + \int_{-\infty}^{+\infty} G_R(t-t'; \mathbf{k}) F_{\mathbf{k}}(t')\,dt', \tag{3.6}$$

where

$$G_R(t;\mathbf{k}) = \frac{i}{2\pi}\lim_{\sigma\to 0^+}\int_{-\infty}^{+\infty}\frac{\exp(-i\omega' t)}{\omega'-\omega_0(k)+i\sigma}\,d\omega'. \tag{3.7}$$

$\xi_{\mathbf{k}}^{(1)}(t)$ is the homogeneous ($F_{\mathbf{k}} = 0$) solution to (3.4):

$$\xi_{\mathbf{k}}^{(1)}(t) = r_{\mathbf{k}}^0 \exp[i\theta_{\mathbf{k}}^0(t)]\exp(i\mu_{\mathbf{k}}), \tag{3.8}$$

where

$$\theta_{\mathbf{k}}^0(t) = -\omega_0(k)\,t \tag{3.9}$$

and $\mu_{\mathbf{k}}$ is a time-independent phase.

A solution to (3.6) can be found by iteration. We expand $\xi_{\mathbf{k}}(t)$ in a series of functions $\xi_{\mathbf{k}}^{(n)}(t)$ which are of nth order in the unperturbed amplitude $r_{\mathbf{k}}^0$:

$$\xi_{\mathbf{k}}(t) = \xi_{\mathbf{k}}^{(1)}(t)+\xi_{\mathbf{k}}^{(2)}(t)+\xi_{\mathbf{k}}^{(3)}(t)+\ldots. \tag{3.10}$$

The function $\xi_{\mathbf{k}}^{(2)}(t)$ is given by

$$\xi_{\mathbf{k}}^{(2)}(t) = \int_{-\infty}^{+\infty} G_R(t-t';\mathbf{k})\,F_{\mathbf{k}}^{(2)}(t')\,dt', \tag{3.11}$$

where $F_{\mathbf{k}}^{(2)}$ is found by substituting (3.10) into the expression for $F_{\mathbf{k}}$ and keeping terms of second order. Performing the integration in (3.11), we obtain

$$\xi_{\mathbf{k}}^{(2)}(t) = \lim_{\sigma\to 0^+}\left(\frac{2\pi}{L}\right)\Sigma\,[\xi_{\mathbf{k}_1}^{(1)}\xi_{\mathbf{k}_2}^{(1)}M_1(\mathbf{k}_1,\mathbf{k}_2)+\xi_{-\mathbf{k}_1}^{(1)*}\xi_{-\mathbf{k}_2}^{(1)*}M_2(\mathbf{k}_1,\mathbf{k}_2)+\xi_{\mathbf{k}_1}^{(1)}\xi_{-\mathbf{k}_2}^{(1)*}$$
$$\times M_3(\mathbf{k}_1,\mathbf{k}_2)+\xi_{-\mathbf{k}_1}^{(1)*}\xi_{\mathbf{k}_2}^{(1)}M_4(\mathbf{k}_1,\mathbf{k}_2)]\,\delta^{(2)}. \tag{3.12}$$

Expressions for $M_1(\mathbf{k}_1,\mathbf{k}_2)$, $M_2(\mathbf{k}_1,\mathbf{k}_2)$, $M_3(\mathbf{k}_1,\mathbf{k}_2)$ and $M_4(\mathbf{k}_1,\mathbf{k}_2)$ are given in appendix A. While further iterations could produce expressions for $\xi_{\mathbf{k}}^{(3)}$, $\xi_{\mathbf{k}}^{(4)}$, etc., they will not be required in this paper.

For certain combinations of discrete modes the frequency denominators in (3.12) may become zero. If these combinations are present, (3.12) has infinite terms, and the iterative perturbation method we have employed fails to approximate $\xi_{\mathbf{k}}(t)$. In this situation, a solution of (3.4) can, in principle, be found by solving a set of coupled amplitude and phase equations for the relevant modes (Hsu 1963). Benney (1962) and McGoldrick (1965) have investigated examples of such equations. In the case of a continuous spectrum of modes, which will be discussed in § 4, the singularities are integrable and do not appear in any observable quantities.

4. Frequency perturbations and modal decay rates for gravity–capillary waves

If we let

$$\xi_{\mathbf{k}}(t) = r_{\mathbf{k}}(t)\exp[-i\theta_{\mathbf{k}}(t)], \tag{4.1}$$

where $r_{\mathbf{k}}(t)$ and $\theta_{\mathbf{k}}(t)$ are real, then (3.4) implies that

$$\frac{d\theta_{\mathbf{k}}(t)}{dt} = \omega_0(k)-\frac{1}{|\xi_{\mathbf{k}}(t)|^2}\,\mathrm{Im}\,[F_{\mathbf{k}}(t)\,\xi_{\mathbf{k}}^*(t)] \tag{4.2}$$

and

$$\frac{1}{r_{\mathbf{k}}(t)}\frac{dr_{\mathbf{k}}(t)}{dt} = +\frac{1}{|\xi_{\mathbf{k}}(t)|^2}\,\mathrm{Re}\,[F_{\mathbf{k}}(t)\,\xi_{\mathbf{k}}^*(t)]. \tag{4.3}$$

Since $\theta_{\mathbf{k}}(t)$ is the negative of the instantaneous phase of $\xi_{\mathbf{k}}(t)$, its time derivative, which is given by (4.2), is the instantaneous frequency of the **k**th mode. Equation (4.2) shows that this instantaneous frequency differs from the unperturbed frequency $\omega_0(k)$ by an amount

$$\Delta\omega_{\mathbf{k}}(t) = -\frac{1}{|\xi_{\mathbf{k}}(t)|^2}\operatorname{Im}[F_{\mathbf{k}}(t)\,\xi_{\mathbf{k}}^*(t)] \tag{4.4}$$

that depends on the nonlinear term $F_{\mathbf{k}}(t)$. The frequency perturbation $\Delta\omega_{\mathbf{k}}(t)$ implies that the phase speed of the **k**th mode changes by an amount

$$\Delta c_{\mathbf{k}}(t) = \Delta\omega_{\mathbf{k}}(t)/k. \tag{4.5}$$

Formulae for $\Delta c_{\mathbf{k}}(t)$ have been calculated by Longuet-Higgins & Phillips (1962) for three interacting gravity wave modes.

The lowest-order expression for $\Delta\omega_{\mathbf{k}}(t)$ that has a non-zero average is

$$\Delta\omega_{\mathbf{k}}^{(2)}(t) = -\frac{1}{|\xi_{\mathbf{k}}^{(1)}(t)|^2}\operatorname{Im}[F_{\mathbf{k}}^{(2)}(t)\,\xi_{\mathbf{k}}^{(2)*}(t) + F_{\mathbf{k}}^{(3)}(t)\,\xi_{\mathbf{k}}^{(1)*}(t)]; \tag{4.6}$$

$F_{\mathbf{k}}^{(3)}(t)$ is found by substituting (3.10) into (2.12) and (2.15) and retaining only terms of third order in the amplitudes $\xi_{\mathbf{k}}^{(1)}(t)$.

The lowest-order expression for the instantaneous modal decay rate that has a non-zero average is

$$\lambda_{\mathbf{k}}^{(2)}(t) = -\frac{1}{r_{\mathbf{k}}(t)}\frac{dr_{\mathbf{k}}(t)}{dt} = -\frac{1}{|\xi_{\mathbf{k}}^{(1)}(t)|^2}\operatorname{Re}[F_{\mathbf{k}}^{(2)}(t)\,\xi_{\mathbf{k}}^{(2)*}(t) + F_{\mathbf{k}}^{(3)}(t)\,\xi_{\mathbf{k}}^{(1)*}(t)]. \tag{4.7}$$

The fact that the frequency perturbation $\Delta\omega_{\mathbf{k}}^{(2)}(t)$ and the decay rate $\lambda_{\mathbf{k}}^{(2)}(t)$ are related by being the imaginary and real parts, respectively, of the quantity $F_{\mathbf{k}}\,\xi_k^*/|\xi_k|^2$ is a well-known general property of systems of coupled modes (see, for example Goldberger & Watson 1964, chap. 8).

Since wind-driven ocean waves are stochastic in character (see, for example, Phillips 1969, chap. 4) only mean values of variables are observationally significant. From (4.6) and (4.7) we see that the mean values of $\Delta\omega_{\mathbf{k}}^{(2)}(t)$ and $\lambda_{\mathbf{k}}^{(2)}(t)$ depend on the stochastic properties of the unperturbed amplitude $\xi_{\mathbf{k}}^{(1)}(t)$. An assumption that appears to fit the known properties of wind-driven ocean waves in equilibrium is that the amplitudes $r_{\mathbf{k}}^{(0)}$ of the $\xi_{\mathbf{k}}^{(1)}(t)$ are non-stochastic constants which depend on **k** and that the random phases $\mu_{\mathbf{k}}$ are statistically independent and uniformly distributed from 0 to 2π (Longuet-Higgins 1957; Cartwright 1962). It is a standard result (Rayleigh 1880; Chandrasekhar 1943) that these assumptions imply that the unperturbed surface

$$\eta^{(1)}(x,y,t) = \left(\frac{2\pi}{L}\right)\sum_{\mathbf{k}}[\xi_{\mathbf{k}}^{(1)}(t) + \xi_{-\mathbf{k}}^{(1)*}(t)]\,e^{i\mathbf{k}.\mathbf{r}} \tag{4.8}$$

is, in the limit $L\to\infty$, a Gaussian random process with a spectral density $\psi(\mathbf{k})$ related to $r_{\mathbf{k}}^{(0)}$ by

$$\psi(\mathbf{k}) = 2r_{\mathbf{k}}^{(0)2}. \tag{4.9}$$

The probability distribution of the lowest-order wave height $\eta^{(1)}(x,y,t)$ is then given by the Gaussian distribution

$$P[\eta^{(1)}(x,y,t)] = (2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-\eta^{(1)}(x,y,t)^2/2\sigma^2\}, \tag{4.10}$$

where

$$\sigma^2 = \int d\mathbf{k}\,\psi(\mathbf{k}); \tag{4.11}$$

for wind-driven waves $\psi(\mathbf{k})$ is the wind-wave spectrum. This result is supported by measurements of surface displacement distributions (see Phillips 1969, § 4.10). Higher-order corrections to the wave height $\eta(x, y, t)$ will result in a slight deviation of its probability distribution from the Gaussian distribution (4.10).

The stochastic assumptions discussed above lead to the mean values

$$\langle \Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle = -(r_{\mathbf{k}}^{(0)})^{-2}\,\mathrm{Im}\,\langle F_{\mathbf{k}}^{(2)}(t)\,\xi_{\mathbf{k}}^{(2)*}(t) + F_{\mathbf{k}}^{(3)}(t)\,\xi_{\mathbf{k}}^{(1)}(t)\rangle \tag{4.12}$$

and

$$\langle \lambda_{\mathbf{k}}^{(2)}(t)\rangle = -(r_{\mathbf{k}}^{(0)})^{-2}\,\mathrm{Re}\,\langle F_{\mathbf{k}}^{(2)}(t)\,\xi_{\mathbf{k}}^{(2)*}(t) + F_{\mathbf{k}}^{(3)}(t)\,\xi_{\mathbf{k}}^{(1)*}(t)\rangle, \tag{4.13}$$

where angle brackets denote an average over the random phases $\mu_{\mathbf{k}}$. Performing these averages and taking the limit $L \to \infty$ [recall that $(2\pi/L)^2 \sum_{\mathbf{k}} \to \int d\mathbf{k}$], we obtain after some algebraic manipulations

$$\begin{aligned}
\langle \Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle &= \frac{1}{2}\int d\mathbf{k}_1 \int d\mathbf{k}_2\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\frac{P}{\omega_0(k_1)+\omega_0(k_2)-\omega_0(k)} \\
&\times \left\{\frac{\psi(\mathbf{k}_1)\,\psi(\mathbf{k}_2)}{\psi(\mathbf{k})}\,\tfrac{1}{2}Q_1(\mathbf{k}_1,\mathbf{k}_2)^2 - \psi(\mathbf{k}_1)\,\tfrac{1}{2}Q_1(\mathbf{k}_1,\mathbf{k}_2)\,Q_3(\mathbf{k},-\mathbf{k}_1)\right. \\
&\left. - \psi(\mathbf{k}_2)\,\tfrac{1}{2}Q_1(\mathbf{k}_1,\mathbf{k}_2)\,Q_3(\mathbf{k},-\mathbf{k}_2)\right\} \\
&- \frac{1}{2}\int d\mathbf{k}_1 \int d\mathbf{k}_2\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,\frac{1}{\omega_0(k_1)+\omega_0(k_2)+\omega_0(k)} \\
&\times \left\{\frac{\psi(\mathbf{k}_1)\,\psi(\mathbf{k}_2)}{\psi(\mathbf{k})}\,\tfrac{1}{2}Q_2(-\mathbf{k}_1,-\mathbf{k}_2)^2 + \psi(\mathbf{k}_1)\,\tfrac{1}{2}Q_2(-\mathbf{k}_1,-\mathbf{k}_2)\,Q_2(-\mathbf{k},-\mathbf{k}_1)\right. \\
&\left. + \psi(\mathbf{k}_2)\,\tfrac{1}{2}Q_2(-\mathbf{k}_1,-\mathbf{k}_2)\,Q_2(-\mathbf{k},-\mathbf{k}_2)\right\} \\
&+ \frac{1}{2}\int d\mathbf{k}_1 \int d\mathbf{k}_2\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\,\frac{P}{\omega_0(k_2)-\omega_0(k_1)-\omega_0(k)} \\
&\times \left\{\frac{\psi(\mathbf{k}_1)\,\psi(\mathbf{k}_2)}{\psi(\mathbf{k})}\,Q_3(\mathbf{k}_2,-\mathbf{k}_1)^2 - \psi(\mathbf{k}_1)\,Q_3(\mathbf{k}_2,-\mathbf{k}_1)\,Q_1(\mathbf{k}_1,\mathbf{k}) + \psi(\mathbf{k}_2)\,Q_3(\mathbf{k}_2,-\mathbf{k}_1)\right. \\
&\left. \times Q_3(\mathbf{k}_2,-\mathbf{k})\right\} + \Delta_R\ [\text{or } \tilde{\Delta}_R,\ \text{as the case may be}],
\end{aligned} \tag{4.14}$$

where P denotes a principal-value integral,

$$\begin{aligned}
\Delta_R = \frac{1}{2}\int d\mathbf{k}_1\,\psi(\mathbf{k}_1)&\left\{\frac{1}{2}k\omega_0(k)\left[1+\frac{\omega_0(k_1)^2}{\omega_0(k)^2}\right][|\mathbf{k}+\mathbf{k}_1|+|\mathbf{k}-\mathbf{k}_1|]\right. \\
&+ 2\omega_0(k_1)\,\mathbf{k}\,.\,\mathbf{k}_1 + \frac{1}{2}\frac{\omega_0(k_1)}{k_1}[-|\mathbf{k}+\mathbf{k}_1|\,(\mathbf{k}\,.\,\mathbf{k}_1 - kk_1) - |\mathbf{k}-\mathbf{k}_1|\,(\mathbf{k}\,.\,\mathbf{k}_1 + kk_1) \\
&\left. + 2k\mathbf{k}\,.\,\mathbf{k}_1] - T\frac{k}{\omega_0(k)}[\tfrac{3}{2}k_1^2k^2 - (\mathbf{k}\times\mathbf{k}_1)^2]\right\}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\tilde{\Delta}_R = \frac{1}{2}\int d\mathbf{k}_1\,\psi(\mathbf{k}_1)&\left\{\omega_0(k)\,k_1^2 + 2\omega_0(k_1)\,\mathbf{k}\,.\,\mathbf{k}_1 + \tfrac{1}{2}\omega_0(k_1)\,[|\mathbf{k}+\mathbf{k}_1|\,(2k-|\mathbf{k}+\mathbf{k}_1|)\right. \\
&- |\mathbf{k}-\mathbf{k}_1|(2k-|\mathbf{k}-\mathbf{k}_1|)] \\
&+ \tfrac{1}{2}\,\omega_0(k)\,[|\mathbf{k}+\mathbf{k}_1|\,(k-|\mathbf{k}+\mathbf{k}_1|) + |\mathbf{k}-\mathbf{k}_1|\,(k-|\mathbf{k}-\mathbf{k}_1|)] \\
&\left. - \frac{1}{2}\frac{\omega_0(k_1)^2}{\omega_0(k)}\,k\,[2k_1 - |\mathbf{k}+\mathbf{k}_1| - |\mathbf{k}-\mathbf{k}_1|] - T\frac{k}{\omega_0(k)}[\tfrac{3}{2}k_1^2k^2 - (\mathbf{k}\times\mathbf{k}_1)^2]\right\}.
\end{aligned} \tag{4.16}$$

Similarly,

$$\begin{aligned}\langle\lambda_{\mathbf{k}}^{(2)}(t)\rangle = \frac{\pi}{2}\int d\mathbf{k}_1\int d\mathbf{k}_2\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta[\omega_0(k)-\omega_0(k_1)-\omega_0(k_2)] \\ \times\left\{\frac{\psi(\mathbf{k}_1)\,\psi(\mathbf{k}_2)}{\psi(\mathbf{k})}\,\tfrac{1}{2}Q_1(\mathbf{k}_1,\mathbf{k}_2)^2-\psi(\mathbf{k}_1)\,\tfrac{1}{2}Q_1(\mathbf{k}_1,\mathbf{k}_2)\,Q_3(\mathbf{k},-\mathbf{k}_1)\right. \\ \left.-\psi(\mathbf{k}_2)\,\tfrac{1}{2}Q_1(\mathbf{k}_1,\mathbf{k}_2)\,Q_3(\mathbf{k},-\mathbf{k}_2)\right\} \\ +\frac{\pi}{2}\int d\mathbf{k}_1\int d\mathbf{k}_2\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\,\delta[\omega_0(k)+\omega_0(k_1)-\omega_0(k_2)] \\ \times\left\{\frac{\psi(\mathbf{k}_1)\,\psi(\mathbf{k}_2)}{\psi(\mathbf{k})}\,Q_3(\mathbf{k}_2,-\mathbf{k}_1)^2-\psi(\mathbf{k}_1)\,Q_3(\mathbf{k}_2,-\mathbf{k}_1)\,Q_1(\mathbf{k}_1,\mathbf{k})\right. \\ \left.+\psi(\mathbf{k}_2)\,Q_3(\mathbf{k}_2,-\mathbf{k}_1)\,Q_3(\mathbf{k}_2,-\mathbf{k})\right\}.\end{aligned} \tag{4.17}$$

Here the function $\psi(\mathbf{k})$ is the unperturbed wave spectrum. The use of Δ_R or $\tilde{\Delta}_R$ depends on which of the pairs of solutions $(A_\mathbf{k}, B_\mathbf{k})$ and $(\tilde{A}_\mathbf{k}, B_\mathbf{k})$ is involved. Expressions for $Q_1(\mathbf{k}_1, \mathbf{k}_2)$, $Q_2(\mathbf{k}_1, \mathbf{k}_2)$ and $Q_3(\mathbf{k}_1, \mathbf{k}_2)$ are given in appendix A.

For a spectrum of waves, (4.14)–(4.16) give the lowest-order frequency perturbation due to the nonlinear terms in the equations of inviscid fluid dynamics. The mean phase-velocity perturbation $\langle\Delta\omega_\mathbf{k}^{(2)}(t)\rangle/k$ is the spectral analogue of the quantity given in equation (2.11) in Longuet-Higgins & Phillips (1962).

Equation (4.17) gives the lowest-order modal decay rate due to nonlinear interactions in a gravity–capillary wave spectrum. For the $z = 0$ pair $(A_\mathbf{k}, B_\mathbf{k})$, it implies the same energy transfer rate as that calculated by Valenzuela & Laing (1972) after a correction has been made for what appears to be a typographical error in their paper.† For gravity waves alone ($T = 0$), (4.17) gives $\langle\lambda_\mathbf{k}^{(2)}(t)\rangle = 0$ since the Dirac delta-functions in the integrands cannot be satisfied. This result is well known (Phillips 1960).

5. Comparison of the $z = 0$ and $z = \eta$ methods

Although both the $z = 0$ perturbation method for $(A_\mathbf{k}, B_\mathbf{k})$ and the $z = \eta$ perturbation method for $(\tilde{A}_\mathbf{k}, B_\mathbf{k})$ may produce, for the same observable quantity, series expansions that eventually converge to the same value, their utility is determined by how rapidly these expansions converge. A crude criterion for rapid convergence is

$$|\langle\Delta\omega_\mathbf{k}^{(2)}(t)\rangle/\omega_0(k)| \ll 1, \tag{5.1}$$

which implies that the lowest-order corrections to the unperturbed motion are small deviations from the unperturbed motion. If (5.1) does not hold, the lowest-order corrections are not meaningful, and higher-order corrections must be calculated until the perturbation expansion begins to converge.

When $\psi(\mathbf{k})$ is Phillips' (1969, chap. 4) spectral function

$$\psi(\mathbf{k}) = \begin{cases}(B/\pi)\,k^{-4}, & \mathbf{k}\,.\,\mathbf{U} > 0, \quad g/U^2 < k \ll (g/T)^{\frac{1}{2}}, \\ 0, & \text{otherwise},\end{cases} \tag{5.2}$$

† See appendix B.

where $\mathbf{U}$ is the wind velocity vector and $B = 4{\cdot}6\times10^{-3}$, we can obtain estimates of $\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle/\omega_0(k)$ for gravity waves $[T=0,\ \omega_0(k)=(gk)^{\frac{1}{2}}]$ and for $\mathbf{k}.\mathbf{U}=kU$ and $k \gg g/U^2$:

$$\frac{\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle}{\omega_0(k)} \simeq -\frac{B}{\pi}\frac{4}{3}\left(\frac{kU^2}{g}\right)^{\frac{3}{2}} \quad \text{from the } z=0 \text{ method} \tag{5.3}$$

and

$$\frac{\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle}{\omega_0(k)} \simeq \frac{B}{\pi}\frac{16}{3}\left(\frac{kU^2}{g}\right)^{\frac{1}{2}} \quad \text{from the } z=\eta \text{ method.} \tag{5.4}$$

If the wind speed is about 30 knots, so that $g/U^2 = 4\times10^{-4}\,\mathrm{cm}^{-1}$, then the above formulae give for $k = 0{\cdot}4\,\mathrm{cm}^{-1}$

$$\frac{\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle}{\omega_0(k)} \simeq -61{\cdot}7 \quad \text{from the } z=0 \text{ method} \tag{5.5}$$

and

$$\frac{\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle}{\omega_0(k)} \simeq +0{\cdot}247 \quad \text{from the } z=\eta \text{ method.} \tag{5.6}$$

This comparison shows that the $z=0$ perturbation method for $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ violates (5.1) by a large margin while the $z=\eta$ method for $\tilde{A}_{\mathbf{k}}$ and $B_{\mathbf{k}}$ satisfies (5.1). Equation (5.5) implies that the Hasselmann (1962) perturbation method produces lowest-order results which by themselves are meaningless.

Frieda R. Boyle has performed a numerical evaluation of $[\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle/\omega_0(k)]_{z=0}$ for $T=0$ using the spectrum (5.2) with $g/U^2 = 4\times10^{-4}\,\mathrm{cm}^{-1}$, which corresponds to a 30 knot wind. Her results appear in table 1 along with values for this quantity calculated from (5.3). Boyle's evaluation shows that (5.1) is violated for $k > 0{\cdot}04\,\mathrm{cm}^{-1}$ and that the accuracy of (5.3) is poor when k is not considerably larger than g/U^2. Values of $[\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle/\omega_0(k)]_{z=0}$ for $k > 0{\cdot}4\,\mathrm{cm}^{-1}$ can be calculated from (5.3), but they begin to be meaningless as $k\to 4\,\mathrm{cm}^{-1}$ because surface tension becomes important. Numerical evaluations of (4.14), (4.16) and (4.17) for $T \neq 0$ are planned.

The failure of the $z=0$ perturbation method to satisfy (5.1) for the spectrum (5.2) is basically due to an unfortunate choice for the unperturbed solution. The unperturbed velocity potential in the $z=0$ method is

$$\phi^{(1)}(x,y,z,t) = \left(\frac{2\pi}{L}\right)\sum_{\mathbf{k}}(-i)\frac{\omega_0(k)}{k}[\xi_{\mathbf{k}}^{(1)}(t)-\xi_{-\mathbf{k}}^{(1)*}(t)]\,e^{kz}\,e^{i\mathbf{k}.\mathbf{r}}. \tag{5.7}$$

This unperturbed solution oscillates so rapidly on $z=\eta(x,y,t)$, because of the factor $\exp(kz)$, that the $z=0$ perturbation method fails to converge quickly and produces large lowest-order corrections. Since the unperturbed potential in the $z=\eta$ perturbation method satisfies a boundary condition on $z=\eta(x,y,t)$ and not on $z=0$, it does not involve the factor $\exp(kz)$ and, consequently, does not oscillate rapidly for any value of z. Because of this well-behaved unperturbed potential, lowest-order corrections calculated with the $z=\eta$ perturbation method are small.

A physical reason for the success of the $z=\eta$ perturbation method and for the failure of the $z=0$ perturbation method can be seen by recognizing that most wave spectra, such as the Phillips spectrum (5.2), describe a situation in which the small amplitude, high wavenumber waves literally ride on the large amplitude, low wavenumber waves. Consequently, the velocity potential of the high wavenumber waves should be defined relative to the surface created by the low wavenumber waves.

k (cm^{-1})	$[\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle/\omega_0(k)]_{z=0}$ Numerical evaluation	Equation (5.3)
0·0004	$-1{\cdot}75\times10^{-2}\pm2\,\%$	$-0{\cdot}2\times10^{-2}$
0·004	$-1{\cdot}00\times10^{-2}\pm1\,\%$	$-6{\cdot}2\times10^{-2}$
0·04	$-1{\cdot}52 \quad \pm1\,\%$	$-1{\cdot}95$
0·4	$-5{\cdot}87\times10^{1}\ \pm1\,\%$	$-6{\cdot}17\times10^{1}$

TABLE 1. Values of $[\langle\Delta\omega_{\mathbf{k}}^{(2)}(t)\rangle/\omega_0(k)]_{z=0}$ for the spectrum (5.2) with $g/U^2 = 4\times10^{-4}\,\mathrm{cm}^{-1}$ and $\mathbf{k}.\mathbf{U} = kU$. The upper integration limit for k is $(g/T)^{\frac{1}{2}}$.

This is accomplished by using $\tilde{A}_{\mathbf{k}}$, which is the Fourier component of the potential defined on the surface $z = \eta$. Using the values of the potential on $z = 0$ to define the Fourier component of the potential of the high wavenumber waves makes little physical sense because the high wavenumber waves ride on a surface that, most of the time, is many of their wavelengths away from $z = 0$.

The above considerations imply that the $z = \eta$ perturbation method should be used for analysing nonlinear interactions in a spectrum of surface waves. The $z = 0$ perturbation method developed by Hasselmann (1962) and used by Valenzuela & Laing (1972) simply converges too slowly to give meaningful lowest-order results over the spectrum of wind-driven waves in equilibrium.

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Appendix A. Expressions for the C, K, M and Q functions

After some algebraic manipulations one can show that

$$C_1(\mathbf{k}_1, \mathbf{k}_2) = \frac{\omega_0(k_1)}{k_1}(\mathbf{k}_1 . \mathbf{k}_2 + k_1^2), \tag{A 1}$$

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{|\mathbf{k}_1+\mathbf{k}_2|}{\omega_0(|\mathbf{k}_1+\mathbf{k}_2|)}\,\omega_0(k_1)^2, \tag{A 2}$$

$$C_3(\mathbf{k}_1, \mathbf{k}_2) = \frac{|\mathbf{k}_1+\mathbf{k}_2|}{\omega_0(|\mathbf{k}_1+\mathbf{k}_2|)}\frac{\omega_0(k_1)\,\omega_0(k_2)}{k_1 k_2}\,\tfrac{1}{2}(\mathbf{k}_1 . \mathbf{k}_2 - k_1 k_2), \tag{A 3}$$

$$C_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \omega_0(k_1)\,(\mathbf{k}_1 . \mathbf{k}_2 + \tfrac{1}{2}k_1^2), \tag{A 4}$$

$$C_5(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|}{\omega_0(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|)}[-\omega_0(k_1)^2|\mathbf{k}_1+\mathbf{k}_2| + \omega_0(k_1)^2\,\tfrac{1}{2}k_1 + T\{-\tfrac{3}{2}(\mathbf{k}_1 . \mathbf{k}_2)\,k_3^2 + (\mathbf{k}_1\times\mathbf{k}_2) . (\mathbf{k}_1\times\mathbf{k}_3)\}], \tag{A 5}$$

$$C_6(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|}{\omega_0(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|)}\frac{\omega_0(k_1)\,\omega_0(k_2)}{k_1 k_2}(\mathbf{k}_1 . \mathbf{k}_2 - k_1 k_2)\,(-\tfrac{1}{2}|\mathbf{k}_1+\mathbf{k}_2| + k_1) \tag{A 6}$$

and

$$\tilde{C}_1(\mathbf{k}_1, \mathbf{k}_2) = \frac{\omega_0(k_1)}{k_1}[\mathbf{k}_1 . \mathbf{k}_2 + k_1(k_1 - |\mathbf{k}_1 + \mathbf{k}_2|)], \tag{A 7}$$

$$\tilde{C}_2(\mathbf{k}_1, \mathbf{k}_2) = 0, \tag{A 8}$$

$$\tilde{C}_3(\mathbf{k}_1, \mathbf{k}_2) = \frac{|\mathbf{k}_1 + \mathbf{k}_2|}{\omega_0(|\mathbf{k}_1 + \mathbf{k}_2|)} \frac{\omega_0(k_1)\,\omega_0(k_2)}{k_1 k_2} \tfrac{1}{2}(\mathbf{k}_1 . \mathbf{k}_2 + k_1 k_2), \tag{A 9}$$

$$\tilde{C}_4(\mathbf{k}_1 \mathbf{k}_2, \mathbf{k}_3) = \omega_0(k_1)[-\mathbf{k}_2 . \mathbf{k}_3 + \tfrac{1}{2}k_1(k_1 - |\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|) + |\mathbf{k}_1 + \mathbf{k}_2|(|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3| - |\mathbf{k}_1 + \mathbf{k}_2|)], \tag{A 10}$$

$$\tilde{C}_5(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|}{\omega_0(|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|)} T[-\tfrac{3}{2}(\mathbf{k}_1 . \mathbf{k}_2) k_3^2 + (\mathbf{k}_1 \times \mathbf{k}_2) . (\mathbf{k}_1 \times \mathbf{k}_3)], \tag{A 11}$$

$$\tilde{C}_6(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|}{\omega_0(|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|)} \omega_0(k_1)\,\omega_0(k_2)\,(k_2 - |\mathbf{k}_2 + \mathbf{k}_3|). \tag{A 12}$$

The K functions are defined by

$$K_1(\mathbf{k}_1, \mathbf{k}_2) = \tfrac{1}{2}C_1(\mathbf{k}_1, \mathbf{k}_2) - \tfrac{1}{2}C_2(\mathbf{k}_1, \mathbf{k}_2) + \tfrac{1}{2}C_3(\mathbf{k}_1, \mathbf{k}_2), \tag{A 13}$$

$$K_2(\mathbf{k}_1, \mathbf{k}_2) = -\tfrac{1}{2}C_1(\mathbf{k}_1, \mathbf{k}_2) - \tfrac{1}{2}C_2(\mathbf{k}_1, \mathbf{k}_2) + \tfrac{1}{2}C_3(\mathbf{k}_1, \mathbf{k}_2), \tag{A 14}$$

$$K_3(\mathbf{k}_1, \mathbf{k}_2) = \tfrac{1}{2}C_1(\mathbf{k}_1, \mathbf{k}_2) - \tfrac{1}{2}C_2(\mathbf{k}_1, \mathbf{k}_2) - \tfrac{1}{2}C_3(\mathbf{k}_1, \mathbf{k}_2), \tag{A 15}$$

$$K_4(\mathbf{k}_1, \mathbf{k}_2) = -\tfrac{1}{2}C_1(\mathbf{k}_1, \mathbf{k}_2) - \tfrac{1}{2}C_2(\mathbf{k}_1, \mathbf{k}_2) - \tfrac{1}{2}C_3(\mathbf{k}_1, \mathbf{k}_2). \tag{A 16}$$

The M functions are defined by

$$M_1(\mathbf{k}_1, \mathbf{k}_2) = \frac{K_1(\mathbf{k}_1, \mathbf{k}_2)}{\omega_0(k_1) + \omega_0(k_2) - \omega_0(k) + i\sigma}, \tag{A 17}$$

$$M_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{K_2(\mathbf{k}_1, \mathbf{k}_2)}{-\omega_0(k_1) - \omega_0(k_2) - \omega_0(k) + i\sigma}, \tag{A 18}$$

$$M_3(\mathbf{k}_1, \mathbf{k}_2) = \frac{K_3(\mathbf{k}_1, \mathbf{k}_2)}{\omega_0(k_1) - \omega_0(k_2) - \omega_0(k) + i\sigma}, \tag{A 19}$$

$$M_4(\mathbf{k}_1, \mathbf{k}_2) = \frac{K_4(\mathbf{k}_1, \mathbf{k}_2)}{-\omega_0(k_1) + \omega_0(k_2) - \omega_0(k) + i\sigma}; \tag{A 20}$$

in these definitions $k = |\mathbf{k}_1 + \mathbf{k}_2|$. The Q functions are defined by

$$Q_1(\mathbf{k}_1, \mathbf{k}_2) = K_1(\mathbf{k}_1, \mathbf{k}_2) + K_1(\mathbf{k}_2, \mathbf{k}_1), \tag{A 21}$$

$$Q_2(\mathbf{k}_1, \mathbf{k}_2) = K_2(\mathbf{k}_1, \mathbf{k}_2) + K_2(\mathbf{k}_2, \mathbf{k}_1), \tag{A 22}$$

$$Q_3(\mathbf{k}_1, \mathbf{k}_2) = K_3(\mathbf{k}_1, \mathbf{k}_2) + K_4(\mathbf{k}_2, \mathbf{k}_1). \tag{A 23}$$

Appendix B. Comparison of (4.17) with Valenzuela & Laing

From appendix A we obtain

$$K_1(\mathbf{k}_1, \mathbf{k}_2) + K_1(\mathbf{k}_2, \mathbf{k}_1) = i\frac{\omega_0(k_1)\,\omega_0(k_2)}{k_1 k_2} \frac{k}{\omega_0(k)^2} D^{+,+}_{\mathbf{k}_1, \mathbf{k}_2}, \tag{B 1}$$

$$K_3(\mathbf{k}, -\mathbf{k}_1) + K_4(-\mathbf{k}_1, \mathbf{k}) = i\frac{\omega_0(k_1)}{k_1\,\omega_0(k)} D^{+,+}_{\mathbf{k}_1, \mathbf{k}_2}, \tag{B 2}$$

$$K_3(\mathbf{k}_2, -\mathbf{k}_1) + K_4(-\mathbf{k}_1, \mathbf{k}_2) = i\frac{\omega_0(k_1)\,\omega_0(k_2)}{k_1 k_2} \frac{k}{\omega_0(k)^2} D^{+,-}_{\mathbf{k}_1, \mathbf{k}_2}, \tag{B 3}$$

where $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and $\omega_0(k) = \omega_0(k_1) + \omega_0(k_2)$, and

$$K_3(\mathbf{k}_2, -\mathbf{k}_1) + K_4(-\mathbf{k}_1, \mathbf{k}_2) = i\,\frac{\omega_0(k_1)\,\omega_0(k_2)}{k_1\,k_2}\,\frac{k}{\omega_0(k)^2}\,D^{+,\,-}_{\mathbf{k}_1,\,-\mathbf{k}_2}, \tag{B 4}$$

$$K_1(\mathbf{k}_1, \mathbf{k}) + K_1(\mathbf{k}, \mathbf{k}_1) = i\,\frac{\omega_0(k_1)}{k_1\,\omega_0(k)}\,D^{+,\,-}_{\mathbf{k}_1,\,-\mathbf{k}_2}, \tag{B 5}$$

$$K_3(\mathbf{k}_2, -\mathbf{k}) + K_4(-\mathbf{k}, \mathbf{k}_2) = i\,\frac{\omega_0(k_2)}{k_2\,\omega_0(k)}\,D^{+,\,-}_{\mathbf{k}_1,\,-\mathbf{k}_2}, \tag{B 6}$$

where $\mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1$ and $\omega_0(k) = \omega_0(k_2) - \omega_0(k_1)$; in (B 1)–(B 6), $D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2}$ is defined by

$$D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = +\frac{i}{2}\left\{(\omega_1 + \omega_2)\,(k_1\,k_2 - \mathbf{k}_1\,.\,\mathbf{k}_2) + \omega_1\,\omega_2(\omega_1 + \omega_2)\left(\frac{k_2}{g + Tk_1^2} + \frac{k_1}{g + Tk_2^2}\right)\right.$$
$$\left. - (g + Tk^2)\left[\frac{\omega_2(k_1^2 + \mathbf{k}_1\,.\,\mathbf{k}_2)}{g + Tk_2^2} + \frac{\omega_1(k_2^2 + \mathbf{k}_1\,.\,\mathbf{k}_2)}{g + Tk_1^2}\right]\right\}, \tag{B 7}$$

where $\omega_1 = s_1\,\omega_0(k_1)$, $\omega_2 = s_2\,\omega_0(k_2)$ and $k = |s_1\,\mathbf{k}_1 + s_2\,\mathbf{k}_2|$.

If we use the expressions

$$F(\mathbf{k}) = k^{-1}\omega_0(k)^2\,\psi(\mathbf{k}) \tag{B 8}$$

and

$$\partial F(\mathbf{k})/\partial t = -2k^{-1}\omega_0(k)^2\,\langle\lambda^{(2)}_{\mathbf{k}}(t)\rangle\,\psi(\mathbf{k}), \tag{B 9}$$

which follow from the definition of $F(\mathbf{k})$ as the lowest-order energy density spectrum [Valenzuela & Laing 1972, equation (3.3)], and substitute the expression for $\langle\lambda^{(2)}_{\mathbf{k}}(t)\rangle$ given by (4.17) with the D functions replacing the K functions according to (B 1)–(*B* 6), then we obtain equation (3.5) of Valenzuela & Laing (1972). Unfortunately, our expression (B 7) is not the same as that derived by Valenzuela & Laing, who obtain [below their (2.12)]

$$D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = \frac{i}{2}\left\{(\omega_1 + \omega_2)\,(k_1\,k_2 - \mathbf{k}_1\,.\,\mathbf{k}_2) + \omega_1\,\omega_2(\omega_1 + \omega_2)\left(\frac{k_1}{g + Tk_1^2} + \frac{k_2}{g + Tk_2^2}\right)\right.$$
$$\left. - (g + Tk^2)\left[\frac{\omega_1(k_1^2 + \mathbf{k}_1\,.\,\mathbf{k}_2)}{g + Tk_1^2} + \frac{\omega_2(k_2^2 + \mathbf{k}_1\,.\,\mathbf{k}_2)}{g + Tk_2^2}\right]\right\}. \tag{B 10}$$

The difference between (B 7) and (B 10) appears to be due to typographical errors in Valenzuela & Laing's paper. As a check on (B 7), we observe that for $T = 0$ it agrees with Hasselmann [1962, equation (4.3)], who derives for an infinite bottom

$$D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = i(\omega_1 + \omega_2)\,(k_1\,k_2 - \mathbf{k}_1\,.\,\mathbf{k}_2). \tag{B 11}$$

Equation (B 10) does not agree with (B 11) for $T = 0$.

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